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Spin Interaction with an Ideal Fermi Gas

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We consider the equilibrium dynamics of a system consisting of a spin interacting with an ideal Fermi gas on the lattice $\mathbb{Z}^\nu$, $\nu \geqslant 3$. We present two examples: when this system is unitarily equivalent to an ideal Fermi gas or to a spin in an ideal Fermi gas without interaction between them.

KEY WORDS: Equilibrium dynamics; quantum spin system; Fermi gas; particle in an ideal gas.

1. INTRODUCTION

This self-contained paper is the third in a series of papers[1,2] devoted to a proof of the isomorphism between locally perturbed dynamics and free dynamics. A particle interacting with an ideal gas can be imagined as a local perturbation of the free system consisting of the particle and the ideal gas without mutual interaction between them. Until now there was only one example—a classical particle interacting with a classical gas on the half-line—where the equivalence with the ideal gas was proven[3] (the methods resemble those of Ref. 2).

Here we consider an ideal Fermi gas on the lattice and a spin (situated, e.g., at the point $0 \in \mathbb{Z}^\nu$). This system (due to experience elaborated in Ref. 1) seems to be the simplest system for revealing the spectral reasons for the existence of equivalence between interacting and free systems.

Our first result (Theorem 4) concerns the case when our system is equivalent to an ideal gas. Here the spin "disappears" in the Fermi sea; this

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situation is similar to that in Ref. 3. We give the spectral explanation of this phenomenon: the eigenvalue becomes the resonance.

The second result (only for quadratic perturbations) concerns the case when our system is equivalent to an ideal gas plus a free quasiparticle.

2. MØLLER MORPHISMS IN CAR-ALGEBRA

Here we put the main result of Ref. 1 into a more general setting.

Let $\mathscr{H}$ be the complex, separable Hilbert space and $\mathfrak{A} = \mathfrak{A}(\mathscr{H})$ the CAR-algebra over $\mathscr{H}$. It is the C^*-algebra with $\mathbb{1}$ generated by $a^*(f)$, $a(f)$, $f \in \mathscr{H}$ satisfying

$$\begin{aligned} a^*(f)\, a(g) + a(g)\, a^*(f) &= (f, g)\mathbb{1} \\ a(f)\, a(g) + a(g)\, a(f) &= 0 \end{aligned} \tag{2.1}$$

We use the convention that (f, g) is linear in f.

For any self-adjoint operator H in $\mathscr{H}$ one can define the "free" dynamics, i.e., the strongly continuous group τ_t of *-automorphisms of $\mathfrak{A}$, by

$$\tau_t(a(f)) = a(e^{itH} f) \tag{2.2}$$

If $V = V^* \in \mathfrak{A}$, then one can define the perturbed dynamics[4]

$$\begin{aligned} \tau_t^V(A) = \tau_t(A) + \sum_{n=1}^{\infty} i^n \int \cdots \int_{t \geqslant S_n \geqslant \cdots \geqslant S_1 \geqslant 0} dS_1 \cdots dS_n \\ \times [\tau_{S_1}(V), \ldots, [\tau_{S_n}(V), \tau_t(A)] \cdots], \qquad A \in \mathfrak{A} \end{aligned} \tag{2.3}$$

The infinitesimal generator of this group is given by

$$\delta_V = \delta_0 + i[V, \cdot\,] \tag{2.4}$$

where δ_0 is the infinitesimal generator of τ_t. We shall consider direct,

$$\gamma_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}(\tau_t(A)) \tag{2.5}$$

and inverse Møller morphisms

$$\bar{\gamma}_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}(\tau_t^V(A)) \tag{2.6}$$

if they exist.

Theorem 1. Let the following conditions hold:

(a) There exists the dense set $\mathscr{H}_0 \subset \mathscr{H}$ such that for any $f, f' \in \mathscr{H}_0$

$$(e^{itH}f, f') \in L_1(-\infty, \infty) \tag{2.7}$$

(b) $\bar{V} = \bar{V}^*$ is defined as the finite sum of monomials

$$a^*(f_1) \cdots a^*(f_m)\, a(g_1) \cdots a(g_n) \tag{2.8}$$

where $n+m$ is even and $f_i, g_j \in \mathscr{H}_0$.

Then there exists $\varepsilon_0 = \varepsilon_0(\bar{V}) > 0$ such that for $V = \varepsilon \bar{V}$, $|\varepsilon| < \varepsilon_0$, Møller morphisms (2.5) and (2.6) exist. It follows that, e.g.,

$$\gamma_+ \bar{\gamma}_+ = \bar{\gamma}_+ \gamma_+ = \mathbb{1}, \qquad \gamma_+ \tau_t = \tau_t^V \gamma_+ \tag{2.9}$$

Proof. The proof is a modification of the proofs of Theorems 1 and 2 in Ref. 1.

To prove the existence of the direct Møller morphism, it is sufficient to prove the existence of the dense subset $\mathfrak{A}_0 \subset \mathfrak{A}$ such that for any $A \in \mathfrak{A}_0$

$$\| [\tau_t(A), V] \| \in L_1(-\infty, \infty) \tag{2.10}$$

Let us choose

$$\mathfrak{A}_0 = \{a^*(f_1) \cdots a^*(f_m)\, a(g_1) \cdots a(g_n),\ m \geqslant 0, n \geqslant 0, f_i, g_j \in \mathscr{H}_0\} \tag{2.11}$$

Then, if $A = a(f)$, we use the formula

$$[a(e^{itH}f), a^*(f_1) \cdots a^*(f_m)\, a(g_1) \cdots a(g_n)] = \sum_{j=1}^{m} (-1)^{j-1} (f_j, e^{itH}f)\, a^*(f_1) \cdots \check{a}^*(f_j) \cdots a^*(f_m)\, a(g_1) \cdots a(g_n) \tag{2.12}$$

where $^\vee$ means the missing of $a^*(f_j)$. We note that (12) is valid only for $m+n$ even.

A similar formula is valid for $A = a^*(f)$. For general $A \in \mathfrak{A}_0$ we use the identity

$$[AB, C] = A[B, C] + [A, C]B \tag{2.13}$$

several times.

To prove the existence of the inverse Møller morphism, it is sufficient to prove that for any $A \in \mathfrak{A}_0$

$$\|[A, \tau_t^V(V)]\| \in L_1(0, \infty)$$

Again we restrict ourselves to the case $A = a(f)$. Then

$$\int_0^\infty \|[A, \tau_t^V(V)]\| \, dt \leqslant \int_0^\infty \|[A, \tau_t(V)]\| \, dt + \sum_{k=1}^\infty \int_{0 \leqslant S_1 \leqslant \cdots \leqslant S_k \leqslant t < \infty} \cdots \int dS_1 \cdots dS_k \times \|[A, [\tau_{S_1}(V), ..., [\tau_{S_k}(V), \tau_t(V)] \cdots]\| \quad (2.14)$$

Let us put $V = \varepsilon \sum_{j=1}^L V_j$, where

$$V_j = a^*(f_1^j) \cdots a^*(f_{m_j}^j) \, a(g_1^j) \cdots a(g_{n_j}^j) \quad (2.15)$$

Let

$$M = \|f\| + \sum_{j=1}^L \sum_{I_j, J_j} \prod_{u \in I_j, k \in J_j} \|f_i\| \, \|g_k\| \quad (2.16)$$

where the sum is over all $I_j \subseteq \{1, ..., m_j\}$, $J_j \subseteq \{1, ..., n_j\}$; and

$$S_j = \{f_1^j, ..., f_{m_j}^j, g_1^j, ..., g_{n_j}^j\}, \qquad S = \bigcup_{j=1}^L S_j, \qquad |S| = \sum_{j=1}^L (m_j + n_j)$$

Then

$$\|[a(f), ..., [\tau_{S_n}(V), \tau_t(V)] \cdots]\| \leqslant \varepsilon^{n+1} L^{n+1} M^{n+1} \Sigma^{(j)} \Sigma^{(h)} \times |(f, [\exp(iS_{j_0}H)] \, h_0')| \prod_{l=1}^n |([\exp(iS_l H)] \, h_l, [\exp(iS_{j_l})] \, h_l')| \quad (2.17)$$

where the sum $\Sigma^{(h)}$ is over all $|S|^{2n+1}$ ordered sequences h_0', $h_1, h_1', ..., h_n, h_n'$, where $h_i, h_i' \in S$. The sum $\Sigma^{(j)}$ is over all sequences $(j_0, j_1, ..., j_n)$ such that for all $d = 0, 1, ..., n$:

(1) $d < j_d \leqslant n+1$ (we also specify $S_{n+1} = t$).

(2) j_d can be equal to l for any $1 \leqslant l \leqslant n+1$ at most $|S|$ times.

In the remaining part of the proof (if is quite similar to Ref. 1) one uses only the fact that

$$|([\exp(iS_l H)] f, [\exp(iS_{j_l} H)] \, g)| = B(S_l - S_{j_l}) \in L_1(0, \infty) \quad (2.18)$$

3. QUADRATIC PERTURBATIONS

Any self-adjoint, finite sum of quadratic monomials in $\mathcal{A}$ with the conservation of the number of particles can be represented as

$$V = \sum_{i=1}^{n} (\pm)\, a^*(f_i)\, a(f_i) \tag{3.1}$$

up to a constant.

Let us put for some $f_1, \ldots, f_n$

$$P(f_i)\, g = (g, f_i) f_i, \qquad P = \sum_{i=1}^{n} P(f_i) \tag{3.2}$$

Lemma 1. Let V be given by (3.1). Then the dynamics (2.3) is a free one and can be represented as

$$\tau_t^V(a(f)) = a(e^{it(H+P)} f) \tag{3.3}$$

Proof. Easy calculation shows that [see (2.4)]

$$\delta_V(a(g)) = a(i(H+P)\, g)$$

Then (3.3) follows.

We shall not pursue the case when V does not conserve the number of particles.

4. THE MAIN RESULTS

Let $\mathcal{A}(l_2(\mathbb{Z}^\nu))$, $\nu \geqslant 3$, be the CAR-algebra of the lattice Fermi gas and $\mathcal{A}(\mathbb{C})$ be the finite-dimensional CAR-algebra generated by $\{1, b, b^*\}$.

The latter algebra describes the spin and b, b^* satisfy the standard anticommutation relations. The tensor product of these superalgebras (in the sense of superalgebras) is again the CAR-algebra

$$\mathcal{A} = \mathcal{A}(l_2(\mathbb{Z}^\nu)) \otimes \mathcal{A}(\mathbb{C}) = \mathcal{A}(\mathbb{C} \oplus l_2(\mathbb{Z}^\nu))$$

generated by 1, b, b^*, $a(f)$, $a^*(f)$, $f \in l_2(\mathbb{Z}^\nu)$. We denote for convenience

$$b = a(\varphi_0), \qquad \varphi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C} \oplus l_2(\mathbb{Z}^\nu)$$

The free dynamics in $\mathcal{A}$ is defined by

$$\tau_t(a(f)) = a([\exp(it\tilde{H})] f), \qquad \tau_t(b) = [\exp(irt\lambda)]\, b, \quad \lambda \in \mathbb{C}$$

where $\tilde{H} = -\Delta + \mu \mathbb{1}$ is the lattice Laplacian plus constant. Let ω_{sp} be the ground state on $\mathfrak{A}(\mathbb{C})$ and ω_{gas} be the ground or temperature state on $\mathfrak{A}(l_2(\mathbb{Z}^\nu))$, which are equilibrium with respect to the free dynamics.

Then the free Hamiltonian in the GNS representation with respect to $\omega_0 = \omega_{sp} \otimes \omega_{gas}$ is

$$H_0(\lambda) = H_{sp} \otimes \mathbb{1} + \mathbb{1} \otimes H_{gas} \tag{4.1}$$

where

$$H_{sp} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$$

acts in $\mathbb{C}^2$ and H_{gas} is written down explicitely in Ref. 1 and acts in the Fock space or in the Fock-tensor-anti-Fock space.

In the sequel we shall consider the particular cases of the following general situation: let the C^*-algebras $\mathfrak{A}$ and $\mathfrak{A}'$ be given. Let τ_t and τ'_t be the dynamics on them. Let ω be the τ_t-invariant state on $\mathfrak{A}$. Let there exist *-isomorphism $\alpha: \mathfrak{A} \to \mathfrak{A}'$ such that

$$\tau_t = \alpha^{-1} \tau'_t \alpha \tag{4.2}$$

Our main task will be to prove the existence of α in some cases. The following easy proposition allows us to obtain the spectral information from this fact.

Lemma 2. If we denote $\omega' = \omega \circ \alpha^{-1}$ (the state on $\mathfrak{A}'$) then it follows from (4.2) that the GNS Hamiltonians H_ω and $H_{\omega'}$ are unitarily equivalent.

First we consider the general case of isomorphism between two free dynamics.

Let $\mathfrak{A} = \mathfrak{A}(\mathscr{H})$ and $\mathfrak{A}' = \mathfrak{A}(\mathscr{H}')$ be two CAR-algebras, $\tau_t(a(f)) = a(e^{itH} f)$, $f \in \mathscr{H}$, be the free dynamics on $\mathfrak{A}$, and let ω be the quasifree τ_t-invariant state on $\mathfrak{A}$, defined by (see Ref. 1)

$$\omega(a^*(f_m) \cdots a^*(f_1)\, a(g_1) \cdots a(g_n)) = \delta_{nm} \det((Bf_j, g_i)) \tag{4.3}$$

where B is the linear operator in $\mathscr{H}$

$$B = B(H) = \exp(-\beta H)[\mathbb{1} + \exp(-\beta H)]^{-1}, \qquad 0 \leqslant \beta \leqslant \infty \tag{4.4}$$

Let us fix some unitary operator U: $\mathscr{H} \to \mathscr{H}'$ and put

$$H' = UHU^{-1}, \qquad \tau'_t(a(g)) = a(e^{itH'} g), \qquad g \in \mathscr{H}'$$

$$\omega'(a^*(f'_m) \cdots a^*(f'_1)\, a(g'_1) \cdots q(g'_n)) = \delta_{mn} \det((B'f'_j, g'_i)) \tag{4.5}$$

$$f'_j, g'_i \in \mathscr{H}', \qquad B' = UBU^{-1}$$

We get now the isomorphism (4.2) between τ_t and τ_t' if we put

$$\alpha(a(f)) = a(Uf) \tag{4.6}$$

etc.

Now we proceed to a particular example: the quadratic interaction of a spin and an ideal Fermi gas

$$V_q = \varepsilon(b^* a(f) + a^*(f) b) \tag{4.7}$$

for some $f \in l_2(\mathbb{Z}^\nu)$ with finite support.

Then, using Lemma 1, we have

$$\begin{aligned}
\tau_t^{V_q}(a(G)) &= a(e^{it(H+P)}G), \qquad G \in \mathbb{C} \oplus l_2(\mathbb{Z}^\nu) \\
H &= H(\lambda) = \lambda \mathbb{1} \oplus \tilde{H} \\
PG &= \varepsilon(P((0, f) + \varphi_0) - P((0, f)) - P(\varphi_0))G \\
&= \varepsilon((G, \varphi_0)F + (G, F)\varphi_0) \\
F &= (0, f)
\end{aligned} \tag{4.8}$$

Lemma 3. Let f be of finite support on $\mathbb{Z}^\nu$ and its Fourier transform $\tilde{f}$ be not identically zero on any level surface of the function

$$u(k) = \sum_{i=1}^{\nu} 2(1 - \cos k^i) + \mu \in C^\infty(T^\nu)$$

$T = [0, 2\pi)$, $k = (k^1, \ldots, k^\nu)$. Then there exists $\varepsilon_0(V_q) > 0$ such that for $|\varepsilon| < \varepsilon_0$ the operators $H(\lambda) + P$ and $\tilde{H}$ are unitary equivalent if $\lambda \in (\mu, \mu + 4\nu)$, i.e., λ belongs to the interior of the spectrum of $\tilde{H}$.

Lemma 4. If $\lambda \bar{\in} (\mu, \mu + 4\nu)$, then $H(\lambda) + P$ is unitary equivalent to $H(\lambda')$ for some real number λ' such that

$$|\lambda - \lambda'| = O(|\varepsilon|^2)$$

The proofs of these lemmas are given in the Appendix.

Theorems 2 and 3 stated below concern the case of quadratic interactions, and Theorem 4 that of nonquadratic one

Theorem 2. Under the conditions of Lemma 3, there exists an isomorphism α: $\mathfrak{A}(\mathbb{C} \oplus l_2(\mathbb{Z}^\nu)) \to \mathfrak{A}(l_1(\mathbb{Z}^\nu))$ such that

$$\alpha \tau_t^{V_q}(A) = \tau_t(\alpha(A)), \qquad A \in \mathfrak{A}(\mathbb{C} \oplus l_2(\mathbb{Z}^\nu)) \tag{4.9}$$

and τ_t is the free dynamics of the ideal gas.

Proof. Let U be the unitary operator such that

$$U: \mathbb{C} \oplus l_2(\mathbb{Z}^\nu) \to l_2(\mathbb{Z}^\nu), \qquad U(H+P)U^{-1} = \tilde{H}$$

It exists by Lemma 3. Let us put

$$\alpha a(G) = a(UG), \qquad G \in \mathbb{C} \oplus l_2(\mathbb{Z}^\nu) \tag{4.10}$$

Then

$$\begin{aligned} \alpha\tau_t^{V_q}(a(G)) &= \alpha a(\{\exp[it(H+P)]\} G) = a(U\{\exp[it(H+P)]\} G) \\ &= a([\exp(it\tilde{H})]\, UG) = \tau_t(\alpha(a(G))) \end{aligned} \tag{4.11}$$

Quite similarly, we get the following:

Theorem 3. Under the conditions of Lemma 4, there exists an automorphism β of $\mathcal{A}(\mathbb{C} \oplus l_2(\mathbb{Z}^\nu))$ such that

$$\beta\tau_t^{V_q}\beta^{-1} = \tau_t' \tag{4.12}$$

where

$$\tau_t'(a(G)) = a(e^{itH(\lambda')}G) \tag{4.13}$$

Let us consider now the nonquadratic perturbation

$$V = V_q + \varepsilon'\bar{V} \tag{4.14}$$

where $\bar{V} = \bar{V}^*$ is the finite sum of monomials

$$a^*(F_1) \cdots a^*(F_m)\, a(G_1) \cdots a(G_n)$$

with $n+m$ even and F_i, $G_j \in \mathbb{C} \oplus l_2(\mathbb{Z}^\nu)$ having finite support.

Theorem 4. Under the conditions of Lemma 3, there exists $\varepsilon_0' > 0$ such that for $|\varepsilon'| < \varepsilon_0'$ the dynamics τ_t^V is isomorphic to the free dynamics τ_t of the ideal Fermi gas.

Proof. By Theorem 2 it is sufficient to prove the isomorphism of τ_t^V and $\tau_t^{V_q}$. This will follow if we prove the existence of Møller morphisms

$$\gamma_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}^V(\tau_t^{V_q}(A))$$

$$\bar{\gamma}_\pm(A) = \lim_{t \to \pm\infty} \tau_{-t}^{V_q}(\tau_t^V(A))$$

To prove their existence, we shall use Theorem 1 with $\tau_t^{V_q}$ instead of τ_t in it.

By (2.7) it is sufficient to verify that

$$(e^{it(H+P)}F, G) \in L_1(-\infty, \infty) \tag{4.15}$$

for $F = (c_1, f)$ and $G = (c_2, g) \in \mathbb{C} \oplus l_2(\mathbb{Z}^\nu)$ with f, g having finite support. This will be proved in the Appendix (Lemma A4).

APPENDIX. INVESTIGATION OF THE FRIEDRICHS MODEL

Here we give the proofs of Lemmas 4 and 5 and of the condition (4.15). This is reduced of course to the investigation of the operator h in the Hilbert space $\mathbb{C} \oplus L_2(T^\nu)$,

$$h \begin{pmatrix} c \\ g \end{pmatrix} = \begin{pmatrix} \lambda c + \varepsilon \int_{T^\nu} g\bar{\varphi}\, dk \\ \varepsilon c\varphi + ug \end{pmatrix} \tag{A1}$$

where

$$c \in \mathbb{C}, \qquad g \in L_2(T^\nu)$$

$$u = \sum_{i=1}^{\nu} 2(1 - \cos k^i) + \mu, \; k^i \in [0, 2\pi)$$

and φ is the Fourier transform of the function $f \in l_2(\mathbb{Z}^\nu)$. This is the well-known Friedrichs model.[10]

So

$$h = h_0 + \varepsilon V \tag{A2}$$

where

$$h_0 \begin{pmatrix} c \\ g \end{pmatrix} = \begin{pmatrix} \lambda c \\ ug \end{pmatrix} \tag{A3}$$

and the perturbation V has rank 2.

The operator h_0 has an absolutely continuous spectrum on $[\mu, \mu + 4\nu]$, an isolated eigenvalue at the point λ, and no (continuous) singular spectrum.

We shall prove that:

(a) If λ lies outside or on the boundary of $[\mu, \mu + 4\nu]$, then under the perturbation εV with ε sufficiently small the absolutely continuous spectrum does not change, a singular spectrum does not appear, and there is a small shift of the eigenvalue λ.

(b) if $\lambda \in (\mu, \mu + 4\nu)$, then under some conditions on V (or on the function φ) the discrete spectrum dissappears and the continuous spectrum does not change.

Lemmas 4 and 5 follow from (a) and (b). We note first that since εV has rank 2, then it is well known[5] that absolutely continuous parts of the spectrum of h and h_0 are unitarily equivalent.

The Discrete Spectrum of *h*

The eigenvalue of h, if it exists, can be taken as $\binom{0}{\psi}$ or $\binom{1}{\psi}$.

First we consider the case of $\binom{0}{\psi}$ with the eigenvalue λ'. Then we get from (A1)

$$u(k)\,\psi(k) = \lambda'\psi(k) \tag{A4}$$

So $\psi(k) = 0$ a.e. and this case cannot happen.

In the case $h\binom{1}{\psi} = \lambda'\binom{1}{\psi}$ one has

$$\lambda + \varepsilon \int_{T^\nu} \psi\bar{\varphi}\, dk = \lambda' \tag{A5}$$

$$\varepsilon\varphi + u\psi = \lambda'\psi \tag{A6}$$

Then

$$\psi = -\varepsilon\varphi/(u - \lambda') \tag{A7}$$

If $\lambda' \bar{\in} [\mu, \mu + 4\nu]$, then $\psi \in L_2(T^\nu)$ and the substitution of (A7) into (A5) gives the equation

$$\lambda - \varepsilon^2 \int_{T^\nu} \frac{|\varphi|^2}{u - \lambda'}\, dk = \lambda' \tag{A8}$$

If $\lambda \bar{\in} (\mu, \mu + 4\nu)$, it is easy to prove [see the similar proof below for $F(\lambda')$] that Eq. (A8) for small ε has the unique solution λ' and moreover

$$\lambda' \bar{\in} [\mu, \mu + 4\nu], \qquad |\lambda' - \lambda| = O(|\varepsilon|^2)$$

Let $\lambda \in (\mu, \mu + 4\nu)$. Here we need the condition on φ: let φ be a smooth function on T^ν not identically zero on any subset of the form $\{K: u(k) = \text{const}\}$ (later this is called condition A) (e.g., $\varphi \equiv 1$). Then, for $\lambda' \in (\mu, \mu - 4\nu)$,

$$\varepsilon\varphi/(u - \lambda') \bar{\in} L_2(T^\nu)$$

For small ε Eq. (A8) has no solutions. In fact, the function

$$F(\lambda') \stackrel{\text{def}}{\equiv} \int_{T^\nu} \frac{|\varphi|^2}{u - \lambda'}\, dk$$

is smooth on $S=R^1\backslash[\mu, \mu+4\nu]$, $F(\lambda')\to 0$ if $\lambda'\to\pm\infty$, and has finite limits if $\lambda'\to\mu$ from the left or $\lambda'\to\mu+4\nu$ from the right. So $F(\lambda')$ is bounded on S, and choosing ε, we can make $|\varepsilon^2 F(')|$ as small as we want. But

$$|\lambda-\lambda'|\geqslant \min(|\lambda-\mu|, |\lambda-\mu-4\nu|)>0$$

So (A8) has no solutions

The Absence of the (Continuous) Singular Spectrum

Lemma A1. Let either of the following conditions be satisfied:

1. $\lambda\in R^1\backslash(\mu, \mu+4\nu)$, $\varphi\in C^\infty(T^\nu)$
2. $\lambda\in(\mu, \mu+4\nu)$, $\varphi\in C^\infty(T^\nu)$

and φ satisfies condition A.

Then h has no singular spectrum for sufficiently small ε.

To prove this theorem and condition (4.15), we shall use the following well-known facts.

Let $d\mu$ be finite positive measure on R^1 with the support in (a, b) and

$$v_y(x)\equiv v(x, y)=\int P_y(t-x)\, d\mu(t)$$

$$P_y(t-x)=\frac{1}{\pi}\frac{y}{(t-x)^2+y^2}, \qquad y>0$$

Lemma A2. (a) (Ref. 12.) Let $v_y(x)$ tend to $\rho(x)\in L_1(a, b)$ when $y\to +0$ in the $L_1(a, b)$-norm.

Then $\mu(t)$ is absolutely continuous and $d\mu(t)=\rho(t)\, dt$.

(b) (Ref. 11.) If $\mu(t)$ is absolutely continuous and $d\mu(t)=\rho(t)\, dt$, $\rho(t)\in C(R^1)$, $\rho(t)=0$ for $t\bar{\in}[a, b]$.

Then $v_y(x)$ tends to $\rho(x)$ when $y\to +0$ uniformly on $[a, b]$.

Proof of Lemma A1. The essential spectra of h and h_0 coincide. So the singular spectrum of h belongs to $[\mu, \mu+4\nu]$. Let E_x be the spectral family for h. It is sufficient to prove that for a dense subset of vectors F the restriction of the measure $(E_x F, F)$ onto $[\mu, \mu+4\nu]$ is absolutely continuous (with respect to Lebesgue measure). Let us put for $z=x+iy$, $y>0$,

$$R(z)=R_F(z)=((h-z)^{-1}F, F) \tag{A9}$$

F will be specified later; and define measure $\mu(t)$ by

$$v(x, y) \overset{\text{def}}{=} \pi^{-1} \operatorname{Im} R(z) = \pi^{-1} \int \operatorname{Im}(t-z)^{-1}\, d(E_t F, F)$$

$$= \pi^{-1} \int \frac{y}{(t-x)^2 + y}\, d(E_t F, F) \tag{A10}$$

The calculation of the resolvent

$$(h-z)^{-1} \begin{pmatrix} c \\ \psi \end{pmatrix} = \begin{pmatrix} \tilde{c} \\ \tilde{\psi} \end{pmatrix}$$

gives

$$\begin{aligned} \tilde{c} &= \left[c - \varepsilon \int_{T^\nu} \frac{\psi \bar{\varphi}}{u-z}\, dk \right] \Big/ \left[\lambda - z - \varepsilon^2 \int_{T^\nu} \frac{|\varphi|^2}{u-z}\, dk \right] \\ \tilde{\psi} &= \frac{\psi - \varepsilon \tilde{c} \varphi}{u-z} \end{aligned} \tag{A11}$$

Let

$$S = \left\{ F = \begin{pmatrix} c \\ \psi \end{pmatrix} : \psi \in C^\infty(T^\nu) \right\}$$

S is dense in $\mathbb{C} \oplus L_2(T^\nu)$. Let us put for convenience

$$\phi_z(\psi) \equiv \int_{T^\nu} \frac{\psi(k)}{u(k) - z}\, dk \tag{A12}$$

for any $\psi \in C^\infty(T^\nu)$, $\operatorname{Im} z > 0$. Let us fix some $F = \binom{c}{\psi} \in S$, $z = x + iy$, $y > 0$. Then

$$\begin{aligned} R(z) &= \left((h-z)^{-1} \begin{pmatrix} c \\ \psi \end{pmatrix}, \begin{pmatrix} c \\ \psi \end{pmatrix} \right) \\ &= \phi_z(|\psi|^2) + \frac{[c - \varepsilon \phi_z(\psi \bar{\varphi})][\bar{c} - \varepsilon \phi_z(\varphi \bar{\psi})]}{\lambda - z - \varepsilon^2 \phi_z(|\varphi|^2)} \end{aligned} \tag{A13}$$

Lemma A3. For any $\psi \in C^\infty(T^\nu)$, $\nu \geqslant 3$, $z = x + iy$, $y > 0$, and any $x \in [\mu, \mu + 4\nu]$, the following limit exists:

$$\lim_{y \to +0} \phi_{x+iy}(\psi) \overset{\text{def}}{=} \phi_{x+0i}(\psi) \tag{A14}$$

and, moreover, the convergence is uniform on $[\mu, \mu + 4\nu]$ and $F_{x+0i}(\psi) \in C[\mu, \mu + 4\nu]$.

Proof. We can write

$$\phi_z(\psi)=\int_{T^\nu}\frac{\psi(k)\,dk}{u(k)-z}=\int_{-\infty}^{\infty}\frac{J(t)\,dt}{t-z}$$

where $J(t)\equiv J_{\psi,u}(t)$ is the Gelfand–Leray function.[7,8] We shall use the following well-known properties of this function[7,8]:

1. $J(t)\in C^\infty(\mu,\mu+4\nu)$
2. $J(t)\equiv 0,\ t\in R^1\backslash(\mu,\mu+4\nu)$
3. $|J^{(n)}(t)|\sim A_n(t-\mu)^{(\nu/2-1)-n},\ t\to\mu+0$
   $|J^n(t)|\sim B_n(\mu+4\nu-t)^{(\nu/2-1)-n},\ t\to\mu+4\nu-0$

$n=0,1,2,\ldots$, and A_n, B_n are some constants. We have

$$\phi_z(\psi)=\int_{-\infty}^{\infty}\frac{(t-x)\,J(t)\,dt}{(t-x)^2+y^2}+i\int_{-\infty}^{\infty}\frac{yJ(t)\,dt}{(t-x)^2+y^2}\tag{A15}$$

For $\nu\geqslant 3$, $J(t)$ is continuous and it follows from Lemma A2(b) that the second summand in (A15) uniformly on $[\mu,\mu+4\nu]$ tends to $i\pi J(x)$ if $y\to+0$.

Let us prove that the first summand uniformly on R^1 tends to the Hilbert transform of $J(t)$, that is, to

$$GJ(x)=\int_0^{\infty}\frac{J(x+t)-J(x-t)}{t}\,dt\tag{A16}$$

It is easy to see that $J(t)$ satisfies the Lipshitz–Hölder condition for any δ,

$$|J(t+\delta)-J(t)|\leqslant c\,|\delta|^{1/2}\tag{A17}$$

So (A16) is correctly defined. We have

$$v_J(x,y)\overset{\text{def}}{=}\int_{-\infty}^{\infty}\frac{(t-x)\,J(t)\,dt}{(t-x)^2+y^2}=\int_0^{\infty}\frac{t[J(x+t)-J(x-t)]}{t^2+y^2}\,dt$$

Using (A17), we get

$$\begin{aligned}|v_J(x,y)-GJ(x)|&=\left|\int_0^{\infty}\frac{y^2}{t^2+y^2}\left(\frac{JJ(x+t)-J(x-t)}{t}\right)dt\right.\\&=O\left(\int_0^{\infty}\frac{y^2t^{-1/2}}{t^2+y^2}\,dt\right)=o(1)\end{aligned}\tag{A18}$$

So we have proved that $\phi_{x+iy}(\psi)$ uniformly on $[a, b]$ tends to

$$\phi_{x+i0}(\psi) = GJ_\psi(x) + i\pi J_\psi(x) \tag{A19}$$

Since $J_\psi(x)$ satisfies the Lipshitz–Hölder condition (A17), then (see Ref. 13), $GJ_\psi(x)$ satisfies the Lipshitz–Hölder condition with the exponent 1/2. So $\phi_{x+i0}(\psi)$ is continuous. Lemma A3 is proved. So, for any $\psi \in C^\infty(T^\nu)$, $\nu \geqslant 3$,

$$\lim_{y \to +0} R(x+iy) = \phi_{x+i0}(|\psi|^2) + \frac{[c - \varepsilon\phi_{x+i0}(\psi\bar{\varphi})][\bar{c} - \varepsilon\phi_{x+i0}(\varphi\bar{\psi})]}{\lambda - x - \varepsilon^2\phi_{x+i0}(|\varphi|^2)}$$

$$\stackrel{\text{def}}{=} R(x+i0) \tag{A20}$$

exists. Let us prove that this convergence is uniform on any interval $[a, b] \subset (\mu, \mu + 4\nu)$.

In fact, if condition 2 of Lemma A1 is satisfied, then the imaginary part $-\varepsilon^2\pi J_{|\varphi|^2}(x)$ of the denominator in (A20) is not equal to zero. If the condition 1 of Lemma A1 is satisfied, then the real part $\lambda - x - \varepsilon^2 GJ_{|\varphi|^2}(x)$ of the denominator is not equal to zero.

Due to the uniform convergence of $\phi_{x+iy}(\psi)$, $y \to +0$, we have the desired uniform convergence. Lemma A2(a) shows that $(E_x F, F)$ is absolutely continuous on any $[a, b] \subset (\mu, \mu + 4\nu)$, and, moreover,

$$\rho(x) = \pi^{-1} \operatorname{Im} R(x+i0) \tag{A21}$$

Lemma A1 is proved, since μ and $\mu + 4\nu$ are not eigenvalues of h.

The Proof of Condition (4.15)

Lemma A4. Let condition 2 of Lemma A1 be satisfied and ψ_1, $\psi_2 \in C^\infty(T^\nu)$, $\nu \geqslant 3$. Then, for sufficiently small ε,

$$\left(e^{ith}\begin{pmatrix} c_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} c_2 \\ \psi_2 \end{pmatrix}\right) \equiv B(t) \in L_1(-\infty, \infty) \tag{A22}$$

Proof of Lemma A4. By the polarization identity it is sufficient to prove (A22) for

$$\begin{pmatrix} c_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} c_2 \\ \psi_2 \end{pmatrix} = F$$

We have

$$(e^{ish}F, F) = \int e^{isx} \rho(x)\, dx \tag{A23}$$

where $\rho(x)$ is given by (A21) in $[\mu, \mu+4v]$ and is equal to 0 outside of this interval.

To estimate the oscillatory integral (A23), we need the following.

Lemma A5. Let $f(x) \in C^2(0, a]$, $f(0) = 0$, $f(x) \in C[0, a]$, $f^{(n)}(a)=0$, $n=0, 1, 2$. Let us suppose also that for $x \to 0$, (i) $|f'(x)| = O(x^{-\delta})$, (ii) $|f''(x)| = O(x^{-\delta-1})$ for some δ, $0<\delta<1$. Then

$$U(s) = \int_0^a e^{isx} f(x)\,dx \in L_1(-\infty, \infty) \tag{A24}$$

Proof of Lemma A5. Let $\delta_1 \in (0, 1-\delta)$, $h(x) = x^{1-\delta_1} f'(x)$. Integrating by parts, we get

$$U(s) = -\frac{1}{is}\int_0^a e^{isx} f'(x)\,dx = -\frac{1}{is}\int_0^a e^{isx} x^{\delta_1 - 1} h(x)\,dx \tag{A25}$$

It follows from (i) and (ii) that $h(x)$ is of bounded variation on $[0, a]$. So Lemma A5 follows as by Theorem 2 of Ref. 9:

$$|U(s)| = O(|s|^{-1-\delta_1}) \tag{A26}$$

To use this lemma, we must investigate the behavior of ρ', ρ'' in the neighborhood of the points μ and $\mu+4v$. To this end, we shall study the behavior of functions $\phi_{x+i0}(\psi)$, which exist by Lemma A3.

Lemma A6. The function $\phi(x) \equiv \phi_{x+i0}(\psi)$ satisfies the following properties ($n=1, 2$):

(1) $\phi(x) \in C^2(\mu, \mu+4v)$, $\quad \phi(x) \in C[\mu, \mu+4v]$

(2) $|\phi^{(n)}(x)| = O((x-\mu)^{1/2-n})$, $\quad x>\mu$, $\quad x \to \mu$ (A27)

$|\phi^{(n)}(x)| = O((\mu+4v-x)^{1/2-n})$, $\quad x<\mu+4v$, $\quad x \to \mu+4v$

Proof of Lemma A6. Continuity of $\phi(x)$ was proved in Lemma A3. By (A19) we have

$$\phi(x) = GJ(x) + i\pi J(x) \tag{A28}$$

where $J(x) \equiv J_\psi(x)$ is the Gelfand–Leray function.

From the properties 1–3 of $J_\psi(x)$ indicated in the proof of Lemma A3 it follows that ($n \geqslant 0$)

(a) $J(x) \in C^\infty(\mu, \mu+4v)$, $\quad J(x) \in C[\mu, \mu+4v]$

(b) $|J^{(n)}(x)| = O((x-\mu)^{1/2-n})$, $\quad x>\mu$, $\quad x \to \mu$ (A29)

$|J^{(n)}(x)| = O((\mu+4v-x)^{1/2-n})$, $\quad x<\mu+4v$, $\quad x \to \mu+4v$

So it is sufficient to prove that $GJ(x)$ satisfies the properties 1 and 2 of Lemma A6.

Using the C^∞-partition of the identity, one can assume that (we put $\mu = 0$ for convenience) for some $a > 0$

(a′) $J(x) \in C^\infty(0, a)$

(b′) $|J^{(n)}(x)| = O(x^{1/2-n}), \quad x \to +0, \quad n \geqslant 0; \qquad J^{(n)}(a) = 0, \quad n \geqslant 0$ (A30)

Let us prove that $GJ(x)$ is differentiable on $(0, a)$. We shall use the following simple result:

Lemma A7. If the function $xf(x)$ has a derivative in $x_0 \neq 0$, then $f'(x_0)$ exists and

$$f'(x_0) = \frac{1}{x_0} [(xf(x))']|_{x=x_0} - f(x_0)] \tag{A31}$$

Now we use the identity

$$xGJ(x) = G(xJ(x)) - \int_{-\infty}^{\infty} J(t)\, dt \tag{A32}$$

Since $xJ(x) \in C^1(R^1)$ and $xJ(x)$ is identically zero outside $(0, a)$, then $G(xJ(x)) \in C^1(R^1)$ and

$$\begin{aligned} \frac{d}{dx}(G(xJ)(x)) &= G(xJ'(x) + J(x)) \\ &= G(xJ')(x) + GJ(x) \end{aligned} \tag{A33}$$

It follows from (A32) that $xGJ(x) \in C^1(R^1)$. So, by Lemma A7, $GJ(x) \in C^1(0, a)$, and

$$\begin{aligned} (GJ)'(x) &= \frac{1}{x} [(G(xJ))'(x) - GJ(x)] \\ &= \frac{1}{x} G(xJ')(x) = GJ'(x) \end{aligned} \tag{A34}$$

In the second equality we used (A33), and in the third we used (A32) for $J'(x)$.

Let us suppose that for $x \to +0$

$$|(GJ)''(x)| = O(x^{-1/2}) \tag{A35}$$

Since the asymptotic behavior of $xJ'(x)$ and $J(x)$ for $x \to +0$ are the same, there exists $|(G(xJ'))'(x)| = O(x^{-1/2})$ for $x \to +0$.

But, by (A34),

$$x(GJ)'(x) = G(xJ')(x)$$

Then by Lemma A7 there exists $(GJ)''(x)$ for $x \neq 0$,

$$|(GJ)''(x)| = \left| \frac{1}{x} ((G(xJ'))'(x) - (GJ)'(x) \right| = O(x^{-3/2})$$

Now we must prove (A35). We have

$$(GJ)'(x) = GJ'(x) = \int_0^\infty \frac{J'(x+t) - J'(x-t)}{t} \, dt \tag{A36}$$

Let us divide the integral (A36) into three integrals:

$$GJ'(x) = \int_0^{x/2} + \int_{x/2}^{x} + \int_x^\infty \tag{A37}$$

We have the following estimates for these integrals

$$\left| \int_0^{x/2} \frac{J'(x+t) - J'(x-t)}{t} \, dt \right| = O\left(\int_0^{x/2} (x-t)^{-3/2} \, dt \right) = O(x^{-1/2}) \tag{A38}$$

$$\begin{aligned} \left| \int_{x/2}^{x} \frac{J'(x+t) - J'(x-t)}{t} \, dt \right| &\leqslant \frac{2}{x} \int_{x/2}^{x} |J'(x+t) - J'(x-t)| \, dt \\ &\leqslant \frac{4}{x} \int_{x/2}^{x} |J'(x-t)| \, dt = O(x^{-1/2}) \end{aligned} \tag{A39}$$

$$\begin{aligned} \left| \int_x^\infty \frac{J'(x+t) - J'(x-t)}{t} \, dt \right| &= \left| \int_x^\infty \frac{J'(x+t)}{t} \, dt \right| \\ &= O\left(\int_x^\infty \frac{1}{t(x+t)^{1/2}} \, dt \right) = O(x^{-1/2}) \end{aligned} \tag{A40}$$

We changed the variables in the last integral,

$$\int_x^\infty \frac{1}{t(x+t)^{1/2}} \, dt = \frac{1}{x^{1/2}} \int_1^\infty \frac{1}{t(1+t)^{1/2}} \, dt$$

Lemma A6 is proved.

Recall that the density of the measure $(E_x F, F)$, $F = \binom{c}{\psi}$, on $[\mu, \mu + 4\nu]$ is equal to

$$\rho(x) = \frac{1}{\pi} \operatorname{Im} \left(\phi_{x+i0}(|\psi|^2) + \frac{[c - \varepsilon\phi_{x+i0}(\psi\bar{\varphi})][\bar{c} - \varepsilon\phi_{x+i0}(\varphi\bar{\psi})]}{\lambda - x - \varepsilon^2 \phi_{x+i0}(|\varphi|^2)} \right) \tag{A41}$$

Using Lemma A6, it is easy to verify that

$$\begin{aligned}
&(1) \quad \rho(x) \in C^2(\mu, \mu + 4v) \\
&(2) \quad |\rho^{(n)}(x)| = O((x-\mu)^{1/2-n}), \qquad x > \mu, \quad x \to \mu \\
&\qquad |\rho^{n}(x)| = O((\mu + 4v - x)^{1/2-n}), \qquad x < \mu + 4v \\
&\qquad x \to \mu + 4v, \qquad n = 1, 2
\end{aligned} \tag{A42}$$

To end the proof of Lemma A4, one must use the C^{∞}-partition of the identity and Lemma A5.

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